Journal of Engineering Mathematics 22: 189–201 (1988)

On a local predictor of shear instability and the initiation of local turbulence

D. PNUELI and M. FICHMAN
Faculty of Mechanical Engineering, Technion, Israel Institute of Technology, Haifa 32000, Israel

Received 5 November 1986; accepted in revised form 7 March 1988

Abstract. The stability of a local laminar shear flow and its transition into turbulent flow is considered as a local phenomenon. This transition may remain local, in which case the flow field is partially laminar and partially turbulent, or it may spread and make the whole field turbulent. One of the applications of this analysis is the prediction of local heat-convection rates, which are enhanced by local turbulence. Another application is in heart-lung blood pumps, where excessive shear rates are detrimental to red blood cells.

The analysis is Lagrangian, which concentrates on the stability of a fluid particle in maintaining its position in a laminar shear flow. This stability is shown to depend on the magnitude of a non-dimensional parameter, namely the local Reynolds number $\mathrm{Re}_L = ha^2/\nu$ where h is the local shear rate, a is the particle radius and ν is the fluid's kinematic viscosity. It is shown that when, locally, $\mathrm{Re}_L > 530$, the flow is, locally, unstable. The application of this criterion is simple and direct, and in certain cases it can be shown that the resulting unstable flow is indeed turbulent.

Because the analysis relies on an experimental coefficient which has been obtained for a rigid sphere, rather than for a fluid particle, the criterion is introduced at this stage as a conjecture. Several examples are presented which demonstrate the criterion's ability to yield correct predictions for instability.

1. Introduction

The distinction between laminar and turbulent flows in the field of transport phenomena is made at least since the famous Reynolds experiment. Reynolds himself set the magnitude of a non-dimensional parameter, later denoted the Reynolds number, as a criterion for the transition from laminar to turbulent flow in a pipe. Since then, Reynolds numbers were defined for many other flows, and were shown to be reliable predictors of transition in those flows.

Turbulent flows have been treated as random statistical phenomena, and as manifestation of hydrodynamic instability. Statistical, analytical, numerical and experimental methods are used to describe and to correlate these phenomena. It is generally accepted that turbulence starts only after the laminar flow becomes unstable; this is recognized as a necessary condition, although not a sufficient one, because laminar secondary flows may also exist.

This introduction cannot survey the field of turbulent flow. Such a survey would be a paper in its own right, or a chapter in a book. Still one quite general observation is made: there are flows in which parts of the domain are laminar and parts of it are turbulent. Yet no terms such as local instability, or local turbulence, or local turbulence predictor, seem to have been used.

Transport rates are generally higher in turbulent flows than in laminar ones. When only part of the flow field is turbulent the transport rates in this part may become higher. The engineering implication of this possibility is important, e.g., in the local cooling of some electronic equipment, where high transport rates are beneficial, or in the clean-room construction of microcircuits, where very low transport rates are essential. Lately it has been found that shear rates which exceed a certain magnitude cause deterioration of red blood cells, and the flow field in blood pumps must not have even local regions of such damaging flows. It has therefore become quite desirable to define and derive such a local criterion for flow instabilities.

In addition to these practical needs the derived local criterion exhibits two additional apparent advantages which contribute to the motivation for its development:

(i) Clearly, instability and turbulence are not identical phenomena. The current Eulerian procedure to predict the onset of instability requires that the flow field be first solved under the assumption that it is laminar. Then its stability is checked. When found unstable the resulting flow is either turbulent, or a so-called secondary laminar flow, with the decision depending on what other evidence exists. The criterion presented here uses the Lagrangian approach. It also needs the preliminary laminar-flow solution to which it is applied; but then, after instability has been predicted, the criterion may be easily applied again to the resulting turbulent mean-velocity profile. This profile is then checked as to whether it can support itself as a turbulent flow. If it can, then the flow is definitely turbulent.

(ii) The other advantage is that in the Eulerian stability analysis each new flow field must be treated anew, resulting in new characteristic equations. The perturbations used must be shown to constitute a complete series, or the flow cannot be proved stable. The local predictor obtained here from the Lagrangian analysis is the same for all flows, and its application differs from one flow to another on an algebraic level only. The perturbations used in its development do form a complete series. Furthermore, it is applied algebraically in a two-dimensional fashion, in the most sensitive plane, but the phenomena need not be limited to two dimensions; this is because the model of the instability is that of a moving sphere.

Finally, two of the major drawbacks of the local predictor must be mentioned:

(i) It applies to shear instabilities only, and when other instability mechanisms are involved some further development is required.

(ii) Its analysis relies on a coefficient obtained experimentally for a rigid sphere, and not for a "sphere made of fluid".

It is felt that at this stage the results should be presented as a conjecture, until further support for this approach has been accumulated.

As a general observation it is noted that when a flow is checked for instability and shown to be unstable, it is unstable; when shown to be stable, however, it may still be unstable, unless the criterion is proved to exhaust all forms of instability. In this sense any additional criterion is a contribution.

2. Analysis

Let the velocity field in a laminar flow be given by

$$\mathbf{q} = \mathbf{i}U + \mathbf{j}V,$$

where U is a shear flow in the x direction,

$$U = hy, \tag{1}$$

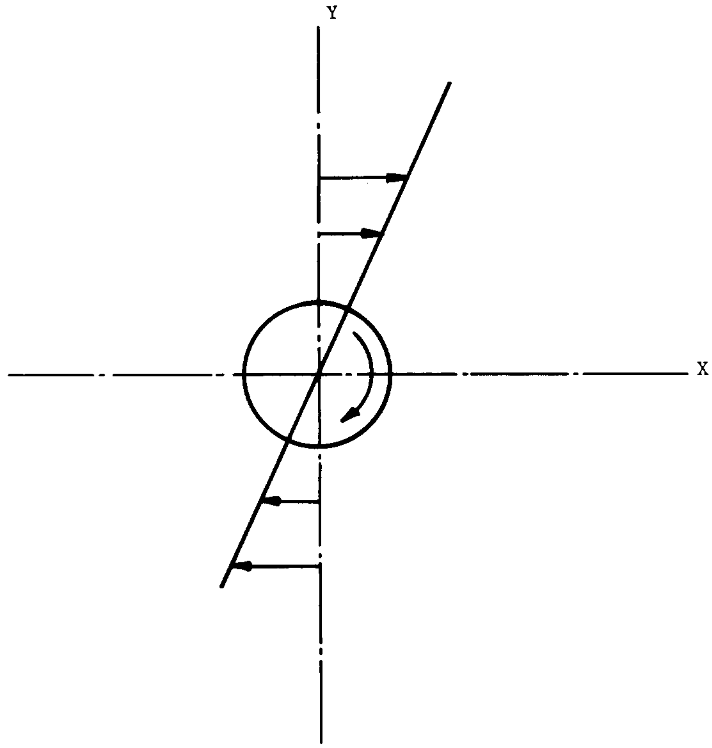

Fig. 1. Fluid particle and coordinates attached to it.

and V is a small sinusoidal perturbation in the y direction, superimposed on the shear flow,

$$V = \varepsilon v(y) \exp\{i(\omega t + kx)\} \tag{2}$$

with $\varepsilon \ll 1$.

Consider a fluid particle in the flow, and let an auxiliary coordinate system move with the particle, staying parallel to the original inertial coordinates. The flow as seen in this auxiliary system is shown in Fig. 1. The small fluid particle rotates with the angular velocity

$$\Omega = \frac{1}{2}\frac{dU}{dy}. \tag{3}$$

Let the coordinates of the fluid particle be X and Y. Because of the perturbation velocity V, the fluid particle changes its Y coordinate. As long as the time mean-value of Y, however, remains the same, the flow is considered stable and remains laminar. Once the value of Y changes monotonically by a finite amount the flow becomes unstable and transition to turbulence occurs. A criterion for this transition is now sought. The equations of motion for the small fluid particle are put in the form

$$\begin{aligned} \frac{d^2X}{dt^2} &= A\left(U - \frac{dX}{dt}\right), \\ \frac{d^2Y}{dt^2} &= A\left(V - \frac{dY}{dt}\right) + L\left(U - \frac{dX}{dt}\right), \end{aligned} \tag{4}$$

where A is "a linear drag coefficient", and L is "a linear lift coefficient". Both A and L are further considered later.

Equations (1) to (4) yield

$$\frac{d^2X}{dt^2} = A\left(hY - \frac{dX}{dt}\right),$$
$$\frac{d^2X}{dt^2} = A[\varepsilon v(Y) \exp\{i(\omega t + kX)\}] + L\left(hY - \frac{dX}{dt}\right), \tag{5}$$

with the initial conditions

$$Y = Y_i, \quad X = 0, \quad \frac{dX}{dt} = U = hY \qquad \text{at } t = 0. \tag{6}$$

The oscillatory perturbation is small, and a solution is sought as a series expansion in ε:

$$X = X_0 + \varepsilon X_1 + \dots, \quad Y = Y_0 + \varepsilon Y_1 + \dots . \tag{7}$$

The zeroth approximation to (5) is

$$\frac{dX_0}{dt} = hY_0, \quad \frac{dY_0}{dt} = 0, \tag{8}$$

with the initial conditions

$$Y_0 = Y_i, \quad X_0 = 0, \qquad \text{at } t = 0, \tag{9}$$

and the zeroth approximation to the solution is

$$Y_0 = Y_i, \quad X_0 = hY_0t. \tag{10}$$

The first-order approximation to (5), using (10), becomes

$$\frac{d^2X_1}{dt^2} + A\frac{dX_1}{dt} - AhY_1 = 0,$$
$$\frac{d^2Y_1}{dt^2} + A\frac{dY_1}{dt} - LhY_1 + L\frac{dX_1}{dt} = Av_0 \exp\{i(\omega t + khY_0t)\}, \tag{11}$$

in which $v_0 = v(Y_0)$. The second equation (11) is rearranged,

$$\frac{dX_1}{dt} = -\frac{1}{L}\frac{d^2Y_1}{dt^2} - \frac{A}{L}\frac{dY_1}{dt} + hY_1 + \frac{A}{L}v_0 \exp(i\bar{\omega}t),$$

and differentiated,

$$\frac{d^2X_1}{dt^2} = -\frac{1}{L}\frac{d^3Y_1}{dt^3} - \frac{A}{L}\frac{d^2Y_1}{dt^2} + h\frac{dY_1}{dt} + i\bar{\omega}\frac{A}{L}v_0 \exp(i\bar{\omega}t),$$

with $\bar{\omega} = \omega + khY_0$, and both expressions are substituted in the first equation (11) to yield

$$\frac{d^3Y_1}{dt^3} + 2A\frac{d^2Y_1}{dt^2} + (A^2 - Lh)\frac{dY_1}{dt} = i\bar{\omega}Av_0 \exp(i\bar{\omega}t) + A^2v_0 \exp(i\bar{\omega}t). \quad (12)$$

The homogeneous part of (12) has a solution of the form

$$Y_{1H} = \exp(Qt)$$

and for

$$Q > 0, \quad \text{or} \quad \text{Real}(Q) > 0,$$

the flow becomes unstable and turbulence starts.

The characteristic equation for Q, from the homogeneous part of (12), is

$$Q^3 + 2AQ^2 + (A^2 - Lh)Q = 0, \quad (13)$$

with the solutions

$$Q_1 = 0, \quad Q_2 = -A - \sqrt{Lh},$$

and the important one

$$Q_3 = -A + \sqrt{Lh}. \quad (14)$$

It is noted that if instead of y-wise oscillations, (2), the perturbations are assumed to oscillate in the x direction, exactly the same analysis leads to another equation (12) which differs from the one obtained here only by its right-hand side. The homogeneous part of the equation remains the same, and therefore the same criterion for stability and transition to turbulence applies.

The criterion for instability is

$$Q_3 > 0,$$

and (14) thus requires

$$Lh > A^2. \tag{15}$$

The criterion has come out in terms of A and L, which therefore merit further consideration.

Assume the fluid particle to *have the shape of a cylinder*, with radius a and length b. The *lift force* on such a cylinder is given by [1, 2]

$$\pi ab \varrho U' \Gamma \tag{16}$$

with ϱ the fluid density, U' the streaming velocity, and Γ the circulation. For the rotating cylinder, (1) and (3) yield

$$\Gamma = 2\pi a \Omega = \pi a h.$$

But, using L as it appears in (4), where LU' is the lift force per total mass of the particle, this becomes

$$L = \pi h. \tag{17}$$

The *drag force* on the cylinder is conveniently expressed as [1, 3]

$$\pi ab C_D \cdot \tfrac{1}{2} \varrho U'^2 \tag{18}$$

with C_D the drag coefficient. But, using A as it appears in (4), where AU' is the drag force per total mass of the cylinder, this becomes

$$A = (U' C_D)/(2a) = (\nu C_D \mathrm{Re}_{U'})/(4a^2) \tag{19}$$

where

$$\mathrm{Re}_{U'} = (2aU')/\nu \tag{20}$$

is a Reynolds number defined for the streaming velocity U' and ν is the kinematic viscosity.

Substitution of (17) and (19) in the criterion for instability, (15), yields a new form for the criterion

$$ha^2/\nu > C_D \mathrm{Re}_{U'}/(4\sqrt{\pi}) = 0.141\ C_D \mathrm{Re}_{U'}. \tag{21}$$

The analysis leading from (16) to (21) can be repeated under the assumption that the fluid particle *has the shape of a sphere*. Equation (16) is still assumed to hold differentially, and is therefore integrated for the sphere. Equation (21) still emerges as the criterion for instability, but with the constant 0.141 replaced by another constant, 0.106.

Now suppose the criterion for instability, (21), is satisfied. This means transition to turbulence. The flow now becomes turbulent, with the turbulent disturbances emerging from the fluid particle. The product

$$C_D \, \mathrm{Re}_{U'}$$

in (21) is, therefore, C_D *at the transition to turbulence* multiplied by $\mathrm{Re}_{U'}$ *at the transition to turbulence**. For the cylindrical shape, this product is 10000 [4], while for the spherical shape it is 5000 [4]. Inserting these values into (21), the final form of the criterion for local instability is

$$ha^2/\nu > 1410 \tag{22}$$

for particles of cylindrical shape, and

$$ha^2/\nu > 530 \tag{23}$$

for spherical particles. Because (23) is less restrictive than (22), it is (23) which must be used as the local criterion for instability.

3. Examples

3.1. Poiseuille flow in a pipe

The velocity profile is

$$u = 2\bar{u}[1 - (r/R)^2],$$

where $\bar{u}$ is the mean velocity, r is the radial coordinate to the center of the fluid particle, and R is the radius of the pipe. The local shear is

$$h = 4\bar{u}r/R^2.$$

The pipe-flow Reynolds number is

$$\mathrm{Re} = 2\bar{u}R/\nu,$$

and the left-hand side of (23) becomes

$$ha^2/\nu = 4\bar{u}r(a/R)^2/\nu = 2(r/R)(a/r)^2 \, \mathrm{Re}.$$

For geometrical reasons, since the particle must be inside the pipe,

$$r + a < R,$$

* At least part of the flow must be turbulent, and $\mathrm{Re}_{U'}$ is taken at 10^4.

Table 1. Transition Reynolds numbers in Poiseuille flow

Assumed particle size					
a/R or a/Y =	0.5	0.4	0.3	0.2	0.1
r/R or y/Y =	0.5	0.6	0.7	0.8	0.9
Pipe Poiseuille flow	2120	2670	4206	8281	29 444
Plane Poiseuille flow	2827	3680	5608	11 041	39 259

and because this particle is assumed not to overlap the center of the pipe,

$$a < R/2.$$

Under these restrictions the highest value attainable by ha^2/ν is for $a = R/2$, with $r = R/2$. Equation (23) now yields

$$\mathrm{Re} = 4 \cdot 530 = 2120.$$

More values of the transition Reynolds number Re, for other assumed values of a/R and r/R, are presented in Table 1.

3.2. Plane Poiseuille flow

An analysis similar to the one for the pipe Poiseuille flow leads to

$$ha^2/\nu = (3/2)(y/Y)(a/Y)^2\ \mathrm{Re},$$

where Y is half the distance between the two plates, and y is measured from the centerline between the plates. For $a = Y/2$, and $y = Y/2$,

$$\mathrm{Re} = (4/3) \cdot 4 \cdot 530 = 2827.$$

Again, more values are presented in Table 1.

Comparison with published data

Experimental values for the transition Reynolds number for the Poiseuille pipe-flow range between Re =_2000 for flows with induced disturbances at the entrance to the pipe, up to Re = 13 000 (and even higher values) for very smooth pipes under very quiescent conditions have been given in [5]. The values presented in Table 1 for this flow, and in particular the smallest value of 2120, seem to compare favorably with these experimental results.

It is interesting to note that this pipe flow is believed to be linearly stable [6], and that it took a fully non-linear theory and three-dimensional perturbations [6] to find its modes of instability, and that by the application of a CRAY computer. There is no contradiction between that non-linearity and the criterion used in this paper, because that non-linearity may well be hidden in the use of the drag coefficients here. However, assuming both methods to be correct, this seems to recommend the use of the one suggested here whenever possible, because of its directness and simplicity.

Computed values for the plane Poiseuille flow [7] indicate the value 2800 for the transition Reynolds number, which is indeed close to the value 2827 obtained here. However, this value has been obtained [7] applying two-dimensional perturbations. The application of three-dimensional perturbations lowered the transition to Re < 500. The implication is that the three-dimensional transition is more involved than a simple shear-instability phenomenon.

3.3. Boundary layer in parallel flow over a flat plate

The boundary-layer thickness in laminar flow may be expressed as [4]

$$\delta/x = 5/\mathrm{Re}_x^{1/2},$$

and the velocity profile may be approximated by [4]

$$u = U[(3/2)(y/\delta) - (1/2)(y/\delta)^3],$$

from which

$$h = (3/2)(U/\delta)[1 - (y/\delta)^2].$$

Hence

$$ha^2/\nu = (15/2)\,\mathrm{Re}_x^{1/2}\,(a/\delta)^2[1 - (y/\delta)^2].$$

For geometrical reasons, $y + a < \delta$ and $y > a$. Under these constrains the highest value attainable by ha^2/ν is for $y = a = \delta/2$, which is

$$ha^2/\nu = (45/32)\,\mathrm{Re}_x^{1/2},$$

and (23) yields the transition criterion $\mathrm{Re}_x = 142\,045$. More values for Re_x are computed under the assumption that $a < \delta/2$, i.e., that the fluid particle is smaller than the thickness of the boundary layer. These values are presented in Table 2.

We choose this example to illustrate another point mentioned in the introduction, that of the decision whether the indicated instability leads to turbulence. To do this, we view turbulence as a state in which the turbulent disturbance velocities, u', v' and w', must be continuously produced, i.e., the mean-velocity profile in the turbulent flow must indicate instability and thus pump energy into these disturbance velocities. We must therefore check now the turbulent boundary-layer flow for stability. A certain transition Reynolds number is expected.

Table 2. Transition Reynolds numbers in boundary-layer flow

Assumed particle size				
a/R or a/Y =	0.5	0.4	0.3	0.2
r/R or y/Y =	0.5	0.6	0.7	0.8
Laminar B.L. flow	142 045	476 244	2.3×10^6	24.1×10^6
Turbulent B.L. flow	260 360	416 310	1.3×10^6	6.9×10^6

Now for

$$\mathrm{Re}_{x_{\mathrm{Laminar}}} < \mathrm{Re}_{x_{\mathrm{Turbulent}}},$$

the laminar flow is indeed unstable, as shown by the analysis. The flow, however, cannot maintain its turbulence, because the mean turbulent velocity does not supply the needed energy. Such a flow may, therefore, exhibit laminar secondary patterns, or change back and forth between the laminar and turbulent modes.

On the other hand, for

$$\mathrm{Re}_{x_{\mathrm{Laminar}}} > \mathrm{Re}_{x_{\mathrm{Turbulent}}},$$

or even for equality of these two numbers, the turbulent flow can maintain itself, and hence the transition is to turbulence.

The boundary-layer thickness in the turbulent flow may be expressed as [4]

$$\delta/x = 0.381/\mathrm{Re}_x^{1/5},$$

and the velocity profile may be approximated by [4]

$$u = U(y/\delta)^{1/7},$$

from which

$$h = (1/7)(U/\delta)(y/\delta)^{-6/7}$$

Hence

$$ha^2/\nu = (0.381/7)\ \mathrm{Re}_x^{4/5}\ (a/\delta)^2(y/\delta)^{-6/7},$$

and again, for $y = a = \delta/2$, (23) yields

$$0.02465\ \mathrm{Re}_x^{4/5} = 530, \quad \text{or} \quad \mathrm{Re}_x = 260\,360.$$

More values for Re_x are computed again under the assumption that $a < \delta/2$, i.e., that the fluid particle is smaller than the thickness of the boundary layer. These values are also presented in Table 2.

Comparison with published data

As pointed out above, transition to turbulence occurs only after the turbulent transition Reynolds number becomes smaller than the laminar one. Application of this condition to Table 2 indicates transition at about $\mathrm{Re}_x = 500\,000$, with $a/\delta = 0.4$. The commonly used values [4] are between 500 000 and 10^7, and again the results obtained here seem quite reasonable.

3.4. Boundary layer in free convection on a vertical flat plate

The free-convection laminar-boundary-layer thickness may be approximated by [8]

$$\delta/x = 3.93\ \mathrm{Pr}^{-1/2}\,(0.952 + \mathrm{Pr})^{1/4}\,\mathrm{Gr}_x^{-1/4},$$

where $\mathrm{Pr} = \nu/\alpha$ is the Prandtl number and $\mathrm{Gr}_x = (g\beta\Delta T x^3)/\nu^2$ is the Grashof number. The velocity profile may be approximated by [7]

$$u = \mathrm{Gr}_x(\nu/4)(\delta^2/x^3)(y/\delta)[1 - (y/\delta)]^2,$$

from which

$$h = \mathrm{Gr}_x^{1/2}\,[3.93\ \mathrm{Pr}^{-1/2}\,(0.952 + \mathrm{Pr})^{1/4}]^2(\nu/4x)(1/\delta)[1 - (y/\delta)][1 - (3y/\delta)].$$

Hence

$$ha^2/\nu = \mathrm{Gr}_x^{1/4}\,(1/4)[3.93\mathrm{Pr}^{-1/2}\,(0.952 + \mathrm{Pr})^{1/4}]^3(a/\delta)^2[1 - (y/\delta)][1 - (3y/\delta)].$$

Inspection of the velocity profile and of this expression indicates that higher values of the expression are obtained near $y/\delta = 2/3$, $a/\delta = 0.04$, which for air, i.e., $\mathrm{Pr} = 0.7$, yields

$$ha^2/\nu = 38\ \mathrm{Gr}_x^{1/4} \cdot 0.053 = 2\ \mathrm{Gr}_x^{1/4}.$$

Now (23) yields $2\ \mathrm{Gr}_x^{1/4} = 520$, $\mathrm{Gr}_x = 5 \times 10^9$.

Comparison with published data

Published experimental data for this case [9] are between about 5×10^8 and more than 10^9. These values are also considered rather close to the one obtained here, more so if the values of $\mathrm{Gr}_x^{1/2}$ are compared instead of the values of the Grashof numbers themselves. However, of all the examples considered here, this is only one where the computed values exceed the experimental ones. When the experimental values are higher, one may wonder whether the necessary perturbations were introduced during the experiments. In a case as the one here, where experiments indicate lower criteria, some additional instability mechanisms may be suspected to be involved. In this case the main suspect appears to be thermal instability [10], superimposed on shear instability. This kind of cooperation has been shown to decrease the stable domain of flows [11].

3.5. Flow around the forward half of a circular cylinder

Conditions are sought under which the flow around the forward half of a circular cylinder remains laminar. The considerations in this example are limited to the forward half of the cylinder in order to avoid the additional complications associated with the possibility of the separation of the boundary layer.

As a rough approximation, the boundary layer around the cylinder is assumed to resemble that on a flat plate, and therefore its length, starting at the forward stagnation point on

Table 3. Computed and measured scales of turbulence in channel flow

Re	$\bar{y}$ =	0.1	0.2	0.3	0.4
30800	h =	126.0	65.5	46.5	31.0
	l_z =	0.67	1.0	1.2	1.4
	l_y =	1.75	2.25	2.4	2.4
	l_a =	0.8–1.6	1.0–2.0	1.25–2.5	1.55–3.1
61600	h =	214.0	143.5	86.0	66.0
	l_z =	0.6	0.8	1.0	1.2
	l_y =	1.3	1.4	1.45	1.5
	l_a =	0.6–1.2	0.72–1.5	0.9–1.8	1.05–2.1

the cylinder, is just

$$x = (\pi/4)D,$$

with D the diameter of the cylinder. For this boundary layer, Example 3.3 gave the transition Reynolds number as $\mathrm{Re}_x = 500\,000$. Hence

$$\mathrm{Re}_D = (4/\pi)\ \mathrm{Re}_x = 636\,600,$$

which is quite within the range of values quoted in the literature [4].

3.6. *Turbulent flow in a two-dimensional channel*

As a final example some details of the shear instability mechanism are illustrated, rather than just the transition to turbulence as in the previous examples. Measurements taken in experiments with turbulent flows in two-dimensional channels [12] are used to determine the scale of the turbulence. The shear rate h is obtained here by numerical differentiation of the measured velocity profiles. Equation (23) is then used to determine the size of the unstable fluid particle, which also gives the order of the size of the turbulence. Values thus obtained, for two Reynolds numbers, are presented in Table 3, which also contains values measured directly in the original experiments [12]. Comparison of these two sets of values in Table 3 again seems to rule in favour of the proposed mechanism.

4. Discussion

A Lagrangian model for the local initiation of shear instability, which may lead to local turbulence, has been presented. Six examples were presented in which this model did yield acceptable numerical values. The importance of the concept of local turbulence and local stability has been stressed in the introduction. With the indication that correct results may be obtained, the model is expected to be useful.

It must be noted, however, that further work is required to better determine the role of the local criterion in instability analysis. Shear instability is but one of many mechanisms capable of inducing instability, and the interaction between several mechanisms is certainly possible. As already stated, the criterion is presented as a conjecture which is considered sufficiently interesting even for just its indication of a simple unified explanation for a certain number of flows.

References

1. L. Prandtl and O.G. Tietjens, *Applied Hydro- and Aeromechanics*, 1st edn., McGraw-Hill Co., N.Y. (1934).
2. S.F. Hoerner and H.V. Borst, *Fluid Dynamics Lift*, pp. 2–11, publ. by L.A. Hoerner, Brick Town, N.J. (1975).
3. S.F. Hoerner, *Fluid Dynamics Drag*, pp. 7–20, publ. by Author, Midland Park, N.J. (1965).
4. H. Schlichting, *Boundary Layer Theory*, 7th edn., McGraw-Hill Co., N.Y. (1979).
5. S. Goldstein, *Modern Developments in Fluid Dynamics*, Vol. 1, Dover Publ., N.Y. (1965).
6. F.T. Smith and R.J. Bodonyi, Amplitude-dependent neutral modes in the Hagen-Poiseuille flow through a circular pipe, *Proc. R. Soc. Lond.* A384 (1982) 463–489.
7. S.A. Orszag and C.K. Kells, Transition to turbulence in plane Poiseuille and plane Couette flow, *J. Fluid Mech.* 96 (1980) 159–205.
8. J.P. Holman, *Heat Transfer*, pp. 266–270, 5th edn., McGraw-Hill Co., N.Y. (1981).
9. E.R.G. Eckert and E. Soehngen, Interferometric studies on the stability and transition to turbulence of a free convection boundary layer, *Proc. Gen. Discuss. Heat Trans. ASME-IME*, London (1951).
10. D. Pnueli, Free convection and its applications, *Keynote Lecture, 7th Int. Conf. Heat Trans.*, München (1982).
11. M. Magen, D. Pnueli and Y. Zvirin, The stability chart of parallel shear flows with double-diffusive processes, *J. Engg. Math.* 19 (1985) 175–187.
12. J. Laufer, Investigation of turbulent flow in two-dimensional channel, *NACA-Rep.* 1053 (1951).